Infinite Cycles and the Graphical Approach to Epistemic Justification

January 22, 2019

Abstract

Recent formal approaches to epistemic justification focus on interpreting beliefs and relations of support between beliefs in graph-theoretic and probabilistic terms. The standard account of the graphical approach to epistemic justification provides graph characterizations for foundationalism, coherentism, and infinitism. In this paper, emerging results from infinite graph theory on infinite cycles are introduced to extend the standard account to include infinite coherentism.

1 Introduction

Over the past decade, great strides have been made in analyzing the structure of epistemic justification mathematically and probabilistically. In particular, the research program beginning with a 2007 paper from Peijnenburg [14] analyses the justification for a given belief in terms of probability theory, motivated primairly by investigating infinite epistemic regresses. One theme resulting from this literature is that the structure of the inferential relations between beliefs matters for determining the epistemic support for a belief. This structure may be expressed in the graph-theoretic terms of vertices, edges, cycles, paths, rays, etc. [3, 4] We refer to this research program as the graphical approach to epistemic justification.

The standard account of the graphical approach provides graph-theoretic characterizations of foundationalism, coherentism, and infinitism where vertices are interpreted as beliefs and edges represent inferential support between two beliefs. Graphs corresponding to foundationalism and coherentism are characterized by every path being bounded in length with the former being acyclic and the latter containing an edge cover in cycle space, i.e., a union of cycles containing every vertex. Graphs corresponding to infinitism are distinct by requiring infinite paths but are acyclic. This categorization of structures of justification is illustrated in Table 1 below. A natural analogue to infintism requires infinite paths and contains an edge cover in cycle space, possibly containing infinite cycles. We refer to this latter justification structure as infinite coherentism. In this paper, emerging results from infinite graph theory on infinite cycles are introduced to extend the standard account to include infinite coherentism.

	Finite Paths	Infinite Paths
Acyclic	Foundationalism	Infinitism
Edge Cover in Cycle Space	Coherentism	Infinite Coherentism

Table 1: Stuctures of Epistemic Justification

This paper proceeds as follows. In Section 2, we introduce the language of graph theory. In Section 3, we consider an approach to infinite cycles developed by Diestel and collaborators. We present graph characterizations for each of foundationalism, coherentism, infinitism, and infinite coherentism in Section 4. Finally, in Section 5, we conclude.

2 Graph Prelimiaries

Let us first introduce terminology for graphs.[1] Let $G = (V, E)$ be a directed graph or digraph with vertex set V and edge set E. We will interpret a vertex as a belief and an edge as a binary relation of justification, in this case as the conditional probability being above a threshold, i.e., $xy \in E$ just in case $P(x \mid y) > \xi$, for some threshold $\xi > 0.5$. Observe that $xy \in E$ does not imply that $yx \in E$. We will refer to $G' = (V, E')$ as the undirected graph underlying the digraph G.

Let $H = (V_H, E_H)$. If $V_H \subset V, E_H \subset E$, then we say that H is a *subgraph* of G. An *edge cover* of G is a subgraph H where every vertex of G is incident to an edge in H. For example, in Figure 1, the subgraph formed by omitting the edge ca is an edge cover of G.

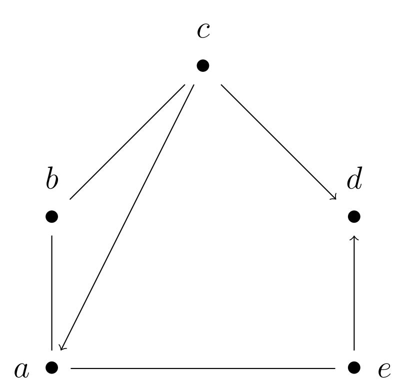

Figure 1: Example Directed Graph with vertices labeled

Let us turn now to properties of graphs and their parts. We say that a *path* is a sequence of edges connecting a sequence of distinct vertices. The *length* of a path is the number of vertices in the path. Let us say that a *finite cycle* $C \subseteq E$ is a path starting and ending at the same vertex where no other vertex is repeated. For example, in Figure 1, the cycle *abca*

[1]The following terminology can be found in any standard graph theory textbook such as [12]

is the edge set $\{ab, bc, ca\}$ where $a, b, c \in V$ and $ab, bc, ca \in E$. We say that a graph is *acyclic* if it contains no cycles.

We call an undirected graph *connected* if there exists a path between any pair of vertices and a directed graph *strongly connected* if there exists a directed path between any pair of vertices. Notice that the undirected graph underlying Figure 1 is connected; however, Figure 1 is not strongly connected, since there exists no path from d to a.

We say that the *degree* of a vertex x is the number of edges incident to x. In the context of a digraph, we define the *in-degree* of a vertex x as the number of incident edges incoming to x and the *out-degree* as the number of incident edges outgoing from x. Consider the example in Figure 1. Vertex d has in-degree of two and out-degree of zero, while vertex a has in-degree of three and out-degree of two.

Consider the set of cycles for a graph G. We say that the *cycle space* of G, denoted $\mathcal{C}(G)$, is the closure of the set of cycles for G by edge disjoint union. Cycle spaces have a well known combinatorial characterization:

Theorem. *Let $H = (V, E)$ be a subgraph of a finite graph G. Then E is an element of the cycle space of G, $\mathcal{C}(G)$, if and only if every vertex of G has even degree in H.*

For infinite graphs, a *ray* is an infinite path with no vertices repeated. A ray has one end point, its root, and continues indefinitely at the other end. Figure 2 below contains a graph of a ray with the root vertex labeled r. A *double ray* consists of two non-intersecting rays beginning from the same vertex. Double rays will prove important for the development of infinte cycles below. Let us define an *in-ray* as a ray in which all edges are only incoming in the direction of the root, and an *out-ray* is a ray in which all edges are only outgoing from the root. We call a ray that is neither in or out mixed.

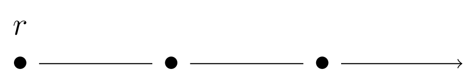

Figure 2: A ray graph with root vertex r

3 Infinite Cycles

In this section, we shall sketch the approach to infinite cycles developed by Diestel and collaborators for locally finite graphs [10].[2] Recall that a finite cycle is a path starting and ending at the same vertex where no other vertex is repeated. Alternatively, we say that a cycle is a connected subgraph where no edge is repeated and each vertex has degree two. We claim that this approach is insufficient to describe infinite cycles. Consider the Double Ray graph in Figure 3. This graph seemingly meets the requirements for a cycle: any vertex can be reached from any other vertex, each vertex has degree two, and no edge is repeated. Surely, this does not accord with the geometric picture of a cycle. As Diestel remarks "... common sense tells us that this can hardly be right: shouldn't cycles be round?" [10]

[2] A locally finite graph is one in which the degree of each vertex is finite. This assumption is made to ensure that the number of edges in the infinite graphs considered throughout are countable.

Figure 3: A Double Ray graph formed by concatenation of two rays at their root vertex

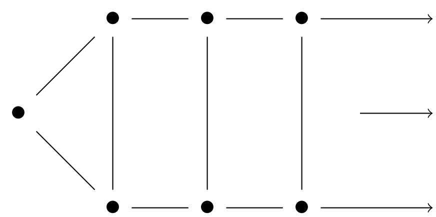

Figure 4: The one-Ladder graph formed from the Double Ray by choosing a middle vertex and adding an edge between the vertices that are equal distance from the middle vertex

Observe the one-Ladder graph in Figure 4. We claim that this graph contains an infinite cycle. Let us turn to developing a framework to make the notion of infinite cycle more precise and thereby validate this claim. A *connected component* of a graph is a connected subgraph that is maximal, i.e., adding any vertex will not preserve connectedness. We define an *end* of a graph to be the set of rays that belong to the same connected component after any finite set of vertices are removed. The Double Ray graph has two ends: one for each concatenated ray. On the other hand, the one-Ladder graph has just one end, since there are infinitely many *rungs* connecting the two rays.

We may fruitfully think of ends as points at infinity.[3] This is presented for the one-Ladder graph in Figure 5. The intuition is that the vertex representing the end is a point that the rays are converging to, so that a cycle can be formed by beginning at the far left vertex, traveling down one ray to the end and back to the far left vertex. This intuition is illustrated in Figure 6. Informally, we say that a graph G is *topologically connected* if there exists a path between any two vertices that is either finite or traverses ends. Observe that both Figure 3 and Figure 4 are topologically connected.

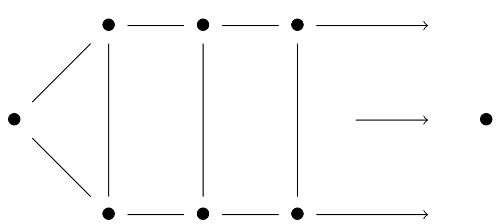

Figure 5: The one-Ladder graph with its end vertex

[3]For a formal development of this idea and the accompanying topology see [8, 9, 11]. Cycles then become topological circles in this space.

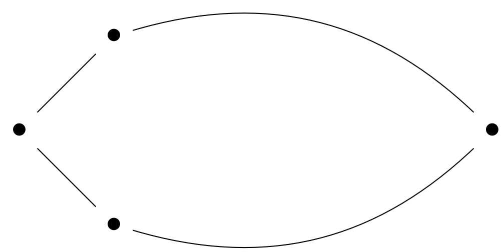

Figure 6: An illustration of the infinite cycle contained in the one-Ladder graph

All that remains is to flesh out the property that excludes the Double Ray as a cycle but includes the one-Ladder: the degree of an end. We say that the *degree of an end* is the maximum number of edge disjoint rays mutually contained in the end. Each end of the Double Ray has degree one, and the end of the one-Ladder has degree two. We rely on a result from Bruhn and Stein [7] to characterize cycles in an infinite graph such as vertex degrees and the degrees of ends to correspond to the account of cycles in finite graphs. We state their result below:

Theorem. *([7], Proposition 3) Let C be a subgraph of a locally finite graph G. Then C is a cycle just in case C is topologically connected and the degree of every vertex and end is two.*

Bruhn and Stein also extend the concept of cycle space to include infinite cycles and provide a combinatorial characterization:

Theorem. *([7], Theorem 4) Let $G = (V, E)$ be a locally fininte graph. Then $E \in \mathcal{C}(G)$ if and only if every vertex and every end of G has even degree.*

Whereas a vertex has an in-degree and out-degree, an end has an in-degree, out-degree, and mixed-degree. Recall that the degree of an end in the underlying graph is the maximum number of edge disjoint rays contained in the end. By classifying those rays as in, out, or mixed, we can obtain the corresponding directed degrees of the end.

Moreover, the directed degrees of an end may have multiple possible configurations. An example will suffice: suppose that the lower ray of the Broken one-Ladder graph in Figure 7 is an out-ray, the upper ray is an in-ray, and the rungs are bi-directional. The end of this graph has degree two; however, that may be made up of one in-ray and one out-ray or two mixed-rays. Rather than being a limitation, this represents the versatility of the end concept; we may choose the configuration that fits our needs.

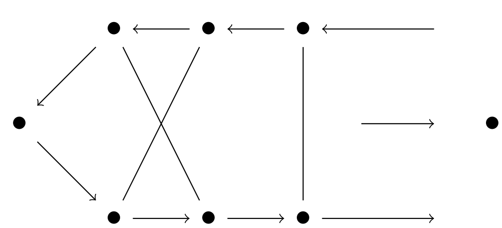

Figure 7: The Broken one-Ladder graph with its end vertex and a particular direction configuration

4 Graphical Approach to Epistemic Justification

In this section, we investigate the structure of epistemic justification using a graph-theoretic approach motivated by the regress problem. The regress problem begins with questioning an agent regarding the justification of a single belief p. Once other beliefs are proffered as evidence for p, we turn to questioning the justification for those beliefs, and so on. We can translate this into graph-theoretic terms by representing p and belief $s_1, s_2, \ldots$ supporting p each with a vertex and a directed edge from s_i to p ranging over all i. Then for each suppporting belief s_i we represent each belief $s_{i,j}$ supporting s_i with a vertex and a directed edge from $s_{i,j}$ to s_i.

Through this iterative process, we obtain the *support graph* for p. Observe that the support graph for p is a subgraph of the graph representing all of an agent's beliefs and relations of support between beliefs.[4] Let us note some properties of support graphs. Let $G = (V, E)$ be a support graph for belief p. By construction, there exists a path in G from v to p for all vertices $v \in V$. For the undirected graph G' underlying G, we have that G' is connected. For any two vertices $x, y \in V$, there exists paths in G' between x, p and y, p. By concatinating these paths, we obtain a path in G' between x, y. Hence, G is connected.

While there is some that we do know about support graphs, importantly, there is much we do not know. We do not know, for instance, whether support graphs are acyclic or contain cycles. Neither do we know if the construction of support graphs terminates in finite or ω steps. In the case of the former, all paths are finite; however, in the case of the latter, at least one path will be infinite. In the sections below, we present four theories of the structure of justification which each take positions on the graph-theoretic properties of support graphs based on epistemicological arguments.

4.1 Foundationalism

Let us begin with the foundationalist approach to epistemic justification. This theory holds that "justification and knowledge are ultimately derivative from a set of basic or foundational elements whose justification does not depend in turn on that of anything else" [6]. Moreover, under foundationalism, justification is not circular [3]. Let $G = (V, E)$ be the support graph for a belief p of a given agent. Foundationalism imposes the following two constraints on G:

(i) The length of the shortest path between v, p for all $v \in V$ is bounded.

(ii) G is acyclic.

Let us consider whether this account of foundationalism appropriately maps our intuitions. Let $b_1, b_2, \ldots$ be an enumeration of the basic beliefs. By construction, there exists a directed path B_i between b_i, p. Since the support graph only extends to b_i, we have that $\lambda(B_i) < \infty$, where λ denotes the length of a path. From constraint (i), we have that $\max_{i \geq 1} \lambda(B_i) < N$ for some natural number N. Observe that for every vertex $v \in V$ there is

[4] Support graphs are the graph-theoretic counterpart to infinite chains and loops discussed by Atkinson and Peijnenburg, Herzberg, and others. See [1, 16, 13, 5] for instances of this approach. On the other hand, see [4] for an approach to developing the graph-theoretic representation for all of an agent's beliefs and relations of support between beliefs.

a path B_j between b_j, p containing v; otherwise, the enumeration of basic beliefs would not be complete. Then the length of the path between v, p is less than N. Clearly, constraint (ii) is justified to exclude circular reasoning.

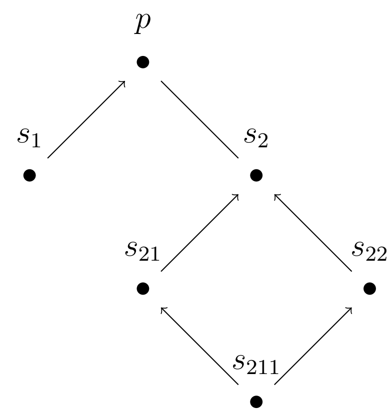

Figure 8: Example Support Graph under Foundationalism

Consider the example in Figure 8. One may be tempted to conclude that G and G' are trees under foundationalism; however, this is not the case. Recall that a tree is an acyclic connected graph. While G is acyclic, it is not necessarily connected. For instance, there is no path between s_1, s_{21} in Figure 8. While G' is connected by construction, it is not necessarily acyclic. In Figure 8, $s_{211}s_{22}s_2s_{21}s_{211}$ is a cycle.

Let us conclude this section by noting another interesting property of foundationalism. Let $H = (V_H, E_H)$ be the support graph for some vertex $v \in V$. Observe that $V_H \subset V, E_H \subset E$, i.e. H is a subgraph of G. Moreover, if $p \notin V_H$ then H is a proper subgraph of G.

4.2 Coherentism

Coherentism is the historical alternative to foundationalism that holds that "beliefs can only be justified by other beliefs...what justifies beliefs is the way they fit together: the fact that they *cohere* with each other" [6]. The boundedness of shortest paths is retained from foundationalism, since it was only recently that infinite epistemic regresses were not met with automatic suspicion. For a support graph $G = (V, E)$, coherentism imposes the following two constraints:

(i) The length of the shortest path between v, p for all $v \in V$ is bounded.

(ii) There exists an edge cover of G in the cycle space of G.

Rather than terminating in basic beliefs, coherentism holds that constructing a support graph will result in the formation of cycles. Constraint (ii), ensures that the support is decomposable into edge-disjoint cycles; it is too restrictive to assume that all vertices belong to a single cycle. Constraint (i) ensures that these cycles are finite.

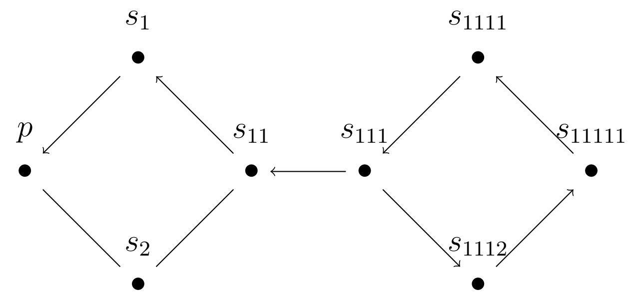

Figure 9: Example Support Graph under Coherentism

Consider the example in Figure 9. Being a finite support graph, constraint (i) is satisfied. Constraint (ii) is satisfied by the edge cover formed from the union of the following two edge-disjoint cycles: $ps_2s_{11}s_1p$ and $s_{111}s_{1112}s_{11111}s_{1111}s_{111}$.

Let C_p denote a cycle in an edge cover containing p.[5] Then any other belief contained in C_p shares the same support graph as p. Furthermore, suppose there exists a directed path between p, v for some $v \in V$. Then v has the same support graph as p. Otherwise, for $u \in V$ such that no directed path exists between p, u, the support graph for u is a proper subgraph of the support graph for p. This property of support graphs under coherentism embodies the notion that these beliefs cohere together but how some beliefs stretch across the web more than others.

4.3 Infinitism

Until the late twentieth century, coherentism had been considered the only reasonable alternative to foundationalism. Infinitism borrows both from foundationalism that justification cannot be circular and from coherentism that only beliefs can justify other beliefs. With this account of epistemic justification, beliefs have infinite chains of support. For a support graph $G = (V, E)$, infinitism imposes the following two constraints:

(i) For each vertex $v \in V$, there is an in-ray to p containing v.

(ii) G is acyclic.

Constraint (ii) is straightforward, since circular reasoning is not permitted. To understand constraint (i), let us contrast infinitism with foundationalism. Both theories of justification impose that G is acyclic. For an acyclic graph, a path is either bounded or unbounded in length. In the case of foundationalism, the boundedness of paths to the root belief p ensures that paths which terminate with basic beliefs do not grow arbitrarily large. In contrast, under infinitism, paths from p do not terminate with basic beliefs; they continue indefinitely as additional beliefs are proffered.

[5]We say a cycle and an edge covering since neither need be unique. In a single edge cover, it may be the case that multiple cycles share p. Similarly, there may be multiple edge covers.

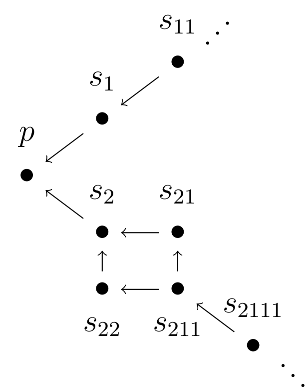

Figure 10: Example Support Graph under Infinitism

Consider the support graph in Figure 10. Observe that every vertex lies on an in-ray to p, and, while there may be cycles in the underlying undirected graph, the directed graph is acyclic. Much like with founationalism, if H is a support graph for $v \in V$ then H is a subgraph of G. However, observe that infinitism is not a generalization of foundationalism in the sense that a support graph under foundationalism does not satisfy the constraints of infinitism.

4.4 Infinite Coherentism

Infinite coherentism is the new structure of justification resulting from the introduction of infinite cycles to support graphs. The intuition for a simple infinite cycle is that the two indefinite reason giving processes go to/from the same end or point at infinity when they are appropriately connected or coherent. For a support graph $G = (V, E)$, infinite imposes only the following constraint:

(i) There exists an edge cover of G in the cycle space of G.

The lone constraint of infinite coherentism ensures that each vertex in the support graph belongs to a cycle though the support graph itself need not be a cycle. Consider the support graph in Figure 11. This is a variant of the one-Ladder graph discussed in Section 3 above. The edge cover for this graph is simply an infinite cycle beginning and ending at p.

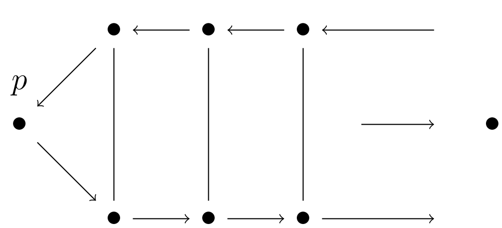

Figure 11: Example Support Graph under Infinite Coherentism

Much like coherentism, a support graph for p has the property of having the same support graph as v if p, v both belong to a cycle or if there exists a directed path p, v. Unlike the relationship between foundationalism and infinitism, infinite coherentism is a generalization of coherentism. This is clear from the given edge cover in cycle space constraint that is shared between the two.

5 Conclusion

By employing the account of infinite cycles as developed by Diestel and collaborators, we have extended the standard account of the graphical approach to epistemic justification to a fourth possibility: infinite coherentism. As with infinitism, this possibility allows for an agent to have infinite paths or chains of beliefs, and, as with coherentism, these beliefs need to cohere through having an edge cover in cycle space.

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