Infinite Cycles in the Graphical Approach to the Regress Problem

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Abstract

Recent work on the regress problem focuses on the conditions under which a proposition has a well defined probability when its justification consists of an infinite linear chain and, more recently, when it consists of an infinite cycle. It is known that these two justification structures lead to distinct results; however, it is not clear how these structures are distinct from a graphical perspective. In this paper, emerging results from infinite graph theory are introduced to clarify the distinction between these structures. While an infinite chain is an in-ray in a directed graph, an infinite cycle is characterized as a topological circle in the space formed by the underlying graph and its ends.

1 Introduction

The regress problem concerns the structure of justification for propositions that we hold, where hold can mean know, believe, etc. Over the past decade, great strides have been made analyzing the structure of justification mathematically and probabilistically, which we refer to as the graphical approach. This research program began with a 2007 paper from Peijnenburg [15] who proved that there exists a case where a proposition has a well defined unconditional probability despite its justification consisting of an infinite[1] linear chain of other propositions, a result with consequences for critics and proponents of infinitism alike.[2]Further results from Peijnenburg and Atkinson and Herzberg elaborated on the conditions where such a well defined unconditional probability exists [2, 12, 13, 16, 17, 18].

Atkinson and Peijnenburg expand their analysis to propositions whose justification structure is either a finite or an infinite cycle of other propositions, relevant for coherentism [1]. They find that the unconditional probability of a proposition whose justification is either a finite or an infinite cycle is almost always well defined; whereas, this is not the case with

[1]Unless mentioned explicitly, infinite is to be regarded as countably infinite.

[2]See the discussion between Carl Ginet and Peter Klein in [20] for two perspecives, against and for, respectively, that were likely surprised by Peijnenburg's result.

infinite linear chains. The authors make a curious observation, where loop is used in place of cycle:

> "At first sight one might think that a nonuniform loop of infinite length cannot really be called a loop...it seems that a real loop differs from an infinite loop.' However, from this it does not follow that, therefore, an infinite loop is in fact an infinite chain. Our investigation shows that such a conclusion would be unwarranted . . . there are exceptional situations in which infinite nonuniform loops and infinite nonuniform chains yield different results" [1].

In this paper, we develop the conceptual difficulties associated with infinite cycles and introduce approaches from infinite graph theory to clarify the distinction between infinite cycles and infinite linear chains. The paper is organized as follows. In the second section, the graphical approach to the regress problem is introduced and its main results are surveyed. In the third section, we illustrate how infinite cycles present conceptual difficulties from the perspective of formal learning theory. In the fourth section, we introduce the topological approach to infinite cycles and derive a condition to distinguish the two. In fifth section, we extend the topological approach to directed graphs which allows us to overcome the hurdles developed in section three. In the sixth, we conclude and offer directions for future work.

2 The Graphical Approach

Let us first introduce a common notation and terminology for graphs.[3]Let $G = (V, E)$ be a directed graph or digraph with vertex set V and edge set E. We will interpret a vertex as a proposition and an edge as a binary relation of justification, in this case as the conditional probability being above a threshold, i.e., $xy \in E$ just in case $P(x \mid y) > \xi$, for some threshold ξ. Observe that $xy \in E$ does not imply that $yx \in E$. Note that these graphs represent the justification structure with respect to a single proposition and not an agent's full set of beliefs. See [3] for an approach in this latter direction. We will refer to $G' = (V, E')$ as the undirected graph underlying the digraph G.

Now that graphs have been introduced, we shall define properties of graphs and their parts. We say that a *path* is a sequence of edges connecting a sequence of vertices. Let us say that a *finite cycle* $C \subseteq E$ is a path starting and ending at the same vertex where no other vertex is repeated. For example, the cycle $abca$ is the edge set $\{ab, bc, ca\}$ where $a, b, c \in V$ and $ab, bc, ca \in E$. We say that the *degree* of a vertex x is the number of edges incident to x. In the context of a digraph, we define the *in-degree* of a vertex x as the incident edges incoming to x and the *out-degree* as the incident edges outgoing from x. Clearly, the degree of a vertex is a sum of its in-degree and out-degree. Finally, we call a graph G *connected* if there exists a path between any pair of vertices in the underlying graph G' and *strongly connected* if there exists a path between any pair in the digraph G.

Specific to infinite graphs, a *ray* is an infinite path with no vertices repeated. A ray has one end point, where it begins, and continues indefinitely at the other end. A *double ray* consists of two non-intersecting rays beginning from the same vertex. We shall introduce the remaining machinery for infinite graphs in section four.

[3]The following terminology can be found in any standard graph theory textbook such as [10]

With this in hand, let us turn to surveying results from the graphical approach, following the presentation in [1]. Let $e_1, e_2, e_3, \dots$ be a sequence of propositions and suppose that we are interested in the unconditional probability of e_1. Observe that we may relate the unconditional probabilities of e_n and another proposition e_{n+1} by the law of total probability:

$$P(e_n) = P(e_n \mid e_{n+1})P(e_{n+1}) + P(e_n \mid \neg e_{n+1})[1 - P(e_{n+1})]. \tag{1}$$

Employing the below substitutions to (1), we obtain

$$P(e_n) = P(e_{n+1})\gamma_n + \beta_n \tag{2}$$

where

$$\alpha_n = P(e_n \mid e_{n+1})$$
$$\beta_n = P(e_n \mid \neg e_{n+1})$$
$$\gamma_n = \alpha_n - \beta_n.$$

Suppose that the structure of the justification for e_1 is a finite linear chain given by $e_1 e_2 e_3 \dots e_{n+1}$. By iterating the recursive formula in (2), we obtain

$$P(e_1) = P(e_{n+1}) \prod_{i=1}^{n} \gamma_i + \sum_{j=1}^{n} \beta_j \prod_{k=1}^{j-1} \gamma_k. \tag{3}$$

Notice that $P(e_1)$ is indeterminate in (3), since its value depends on $P(e_{n+1})$.

Suppose instead that the structure of justification is an infinite linear chain given by $e_1 e_2 e_3 \dots$. We can iterate the recursive formula (2) infinitely many times by taking the limit of (3) as n grows arbitrarily large:

$$P(e_1) = \lim_{n \to \infty} P(e_{n+1}) \prod_{i=1}^{n} \gamma_i + \sum_{j=1}^{\infty} \beta_j \prod_{k=1}^{j-1} \gamma_k. \tag{4}$$

Observe that if $\prod_{i=1}^{n} \gamma_i$ tends to zero as n grows large then $P(e_1)$ is well defined.

Turning now to cycles, suppose that the structure of the justification for e_1 is a finite cycle given by $e_1 e_2 e_3 \dots e_n e_1$. By iterating (2) we obtain

$$P(e_1) = P(e_1) \prod_{i=1}^{n} \gamma_i + \sum_{j=1}^{n} \beta_j \prod_{k=1}^{j-1} \gamma_k \tag{5}$$

and by rearranging terms

$$P(e_1) = \frac{\sum_{j=1}^{n} \beta_j \prod_{k=1}^{j-1} \gamma_k}{1 - \prod_{i=1}^{n} \gamma_i}. \tag{6}$$

In this case, $P(e_1)$ is well defined just in case $\prod_{i=1}^{n} \gamma_i$ is not equal to one. For the infinite analogue in the case of cyclic justification given by the infinite cycle $e_1 e_2 e_3 \dots e_1$, we obtain

$$P(e_1) = \frac{\sum_{j=1}^{\infty} \beta_j \prod_{k=1}^{j-1} \gamma_k}{1 - \prod_{i=1}^{\infty} \gamma_i}. \tag{7}$$

As is the case with (6), $P(e_1)$ is well defined just in case $\prod_{i=1}^{\infty} \gamma_i$ is not equal to one.

We close this section by noting that for the unconditional probability of the proposition of interest to be well defined infinite linear chains require $\prod_{i=1}^{\infty} \gamma_i$ to equal zero while infinite cycles only require $\prod_{i=1}^{\infty} \gamma_i$ not equal one. [4]

3 Challenging Intuitions

The prior section concludes with the result that infinite cycles and infinite linear chains are distinct as justification structures. In this section, we offer a juxtaposition in perspective; a conceptual difficulty is developed regarding identifying infinite cycles combinatorially with the purpose of giving context to the quote from Atkinson and Peijnenburg in the introduction.

Suppose for a given infinite path $\mathcal{P} = p_0 p_1 p_2 \ldots$ in an infinite graph G' we wish to determine whether $\mathcal{P}$ is a ray or contains an infinite cycle. One approach to this question is through formal learning theory, a mathematical framework relating incoming streams of information to hypotheses to understand hypothesis complexity.[5]To generate an incoming stream of information, let us imagine beginning at p_1 and then moving to each subsequent vertex in $\mathcal{P}$. At each vertex, we compute the value for an *encoding function* f as follows:

$$f(p_i) = \begin{cases} 1, & \text{if } p_i = p_j, \text{ for } j < i \\ 0, & \text{otherwise} \end{cases}$$

This encoding function returns zero if the current vertex is unique and returns one if it is repeated.

Let us first consider a simplified case where $\mathcal{P}$ is finite and we wish to determine whether $\mathcal{P}$ contains a cycle. In this case, we simply check all vertices. If $\mathcal{P}$ contains a cycle, then we will see a one after some finite time. On the other hand, if $\mathcal{P}$ does not contain a cycle, then we will see all zeros. We can identify the hypothesis that there is a cycle with the set of sequences of the appropriate length that contain a one; the hypothesis that there is no cycle is the set containing only the all zero sequence.

Increasing the complexity, now suppose that $\mathcal{P}$ is infinite and we wish to determine whether $\mathcal{P}$ contains a finite cycle. Unlike the last case, we cannot check all vertices in finite time. Suppose $\mathcal{P}$ contains a finite cycle. Then after finite amount of time we will see a one. On the other hand, if $\mathcal{P}$ does not contain a finite cycle, then we see only the infinite sequence of all zeros. Note that we are not guaranteed to answer this question in finite time, only in the limit. This formulation is similar to a hypothesis commonly found in the literature: "all swans are white." Suppose we observe an indefinite sequence of swans and form the encoding function that returns zero if the swan is white and one otherwise. The hypothesis "all swans are white" corresponds to the sequence of all zeros: $0000\cdots$. Similarly, observing a non-while swan corresponds to the set of sequences in which there is a one at some observation, following a finite initial sequence of zeros.

The fundamental difficult arises whenever we wish to determine whether the infinite path $\mathcal{P}$ contains an infinite cycle but no finite cycles. Supposing $\mathcal{P}$ contains an infinite cycle, we

[4]See the appendix of [1].

[5]See [14] for the standard reference and [11] for the current direction of the theory.

Figure 1: A Double Ray graph formed by concatenation of two rays at their root vertex

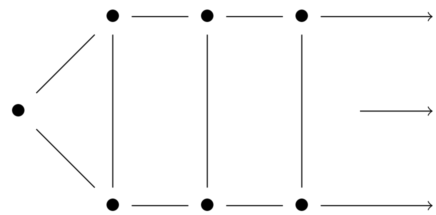

Figure 2: The one-Ladder graph formed from the Double Ray by choosing a middle vertex and adding an edge between the vertices that are equal distance from the middle vertex

will see infinitely many zeros followed by a one: $000\cdots001\cdots$. Otherwise, $\mathcal{P}$ may contain a finite cycle, i.e., the output of the encoding function contains a one after some finite time, or $\mathcal{P}$ may contain no cycles. Observe that knowing that the output converges to one is not helpful since that would permit the existence of a finite cycle. In the language of swans, $\mathcal{P}$ containing an infinite cycle is similar to the hypothesis that after observing infinitely many white swans, we will observe a non-white swan.

To distinguish infinite cycles from infinite linear chains, we must be able to identify infinite cycles. The approach developed in this section illustrates why this is a difficult task.

4 Simple Infinite Cycles

In this section, we shall sketch the approach to infinite cycles developed by Diestel and collaborators. Recall that a finite cycle is a path starting and ending at the same vertex where no other vertex is repeated. Alternatively, we say that a cycle is a connected subgraph where no edge is repeated and each vertex has degree two. We claim that this approach is insufficient to describe infinite cycles. Consider the Double Ray graph in Figure 1. This graph seemingly meets the requirements: any vertex can be reached from any other vertex, each vertex has degree two, and no edge is repeated. Surely, this does not accord with the geometric picture of a cycle. As Diestel remarks ". . . common sense tells us that this can hardly be right: shouldn't cycles be round?" [8]

Observe the one-Ladder graph in Figure 2. We claim that this graph contains an infinite cycle. Let us turn to developing a framework to make the notion of infinite cycle more precise and thereby validate this claim. A *connected component* of a graph is a connected subgraph that is maximal, i.e., adding any vertex will not preserve connectedness. We define an *end* of a graph to be the set of rays that belong to the same connected component after any finite set of vertices are removed. The Double Ray graph has two ends: one for each concatenated ray. On the other hand, the one-Ladder graph has just one end, since there are infinitely many *rungs* connecting the two rays.

We may fruitfully think of ends as points at infinity.[6]This is presented for the one-Ladder graph in Figure 3. The intuition is that the vertex representing the end is a point that the rays are converging to, so that a cycle can be formed by beginning at the far left vertex,

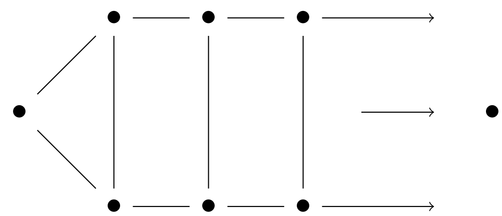

Figure 3: The one-Ladder graph with its end vertex

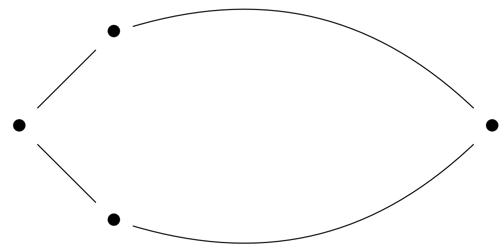

Figure 4: An illustration of the infinite cycle contained in the one-Ladder graph

traveling down one ray to the end and back to the far left vertex. This is illustrated in Figure 4.

All that remains is to flesh out the property that excludes the Double Ray as a cycle but includes the one-Ladder: the degree of an end. We say that the *degree of an end* is the maximum number of edge disjoint rays mutually contained in the end. Each end of the Double Ray has degree one and the end of the one-Ladder has degree two. An infinite cycle is then an infinite edge disjoint path where all vertices and ends have degree two.

5 Infinite Justification Structures via Graphs

To apply the approach developed in the section above to represent infinite justification structures, we must add in the notion of directedness. Recall that in analyzing justification graphically a directed edge is interpreted as inferential support from one proposition to another. Let us define an *in-ray* as a ray in which all edges are only incoming in the direction of the root, and an *out-ray* is a ray in which all edges are only outgoing in the direction of the end. We call a ray that is neither in or out mixed. Whereas a vertex has an in-degree and out-degree, an end has an in-degree, out-degree, and mixed-degree. Recall that the degree of an end in the underlying graph is the maximum number of edge disjoint rays contained in the end. By classifying those rays as in, out, or mixed, we can obtain the corresponding directed degrees of the end.

Moreover, the directed degrees of an end may have multiple possible configurations. An example will suffice: suppose that the lower ray of the Broken one-Ladder graph in Figure 5 is an out-ray, the upper ray is an in-ray, and the rungs go both directions. The end of this graph has degree two; however, that may be made up of one in-ray and one out-ray or

[6]For a formal development of this idea and the accompanying topology see [6, 7]. Cycles then become topological circles in the space.

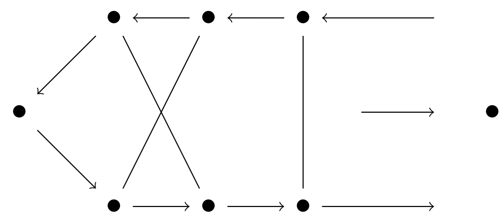

Figure 5: The Broken one-Ladder graph with its end vertex and a particular direction configuration

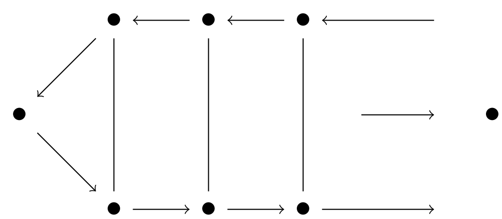

Figure 6: A directed one-Ladder graph with its end vertex and a direction configuration representing the justification structure of an infinite cycle

two mixed-rays. Rather than being a limitation, this represents the versatility of the end concept; we may choose the configuration that fits our needs.

Now that the relevant machinery is developed, let us turn to a graph of interest in Figure 6. This directed one-Ladder captures the intuition of the infinite cycles discussed in section three. Recall that infinite cycles were characterized with the encoding function f as $000\cdots001$. With the directed one-Ladder, we can interpret the initial zeros as traveling along the lower ray, the $\cdots$ as going to and coming from the end, and the trailing zeros as returning to the starting vertex on the upper ray. In contrast, an infinite linear chain is simply an in-ray. Given the degree criteria developed, we are able to distinguish between these justification structures.

We must make note of one caveat in this analysis. To represent infinite cycles of justification as a graph similar to the directed one-Ladder in Figure 6, we must have infinitely many relations between propositions; otherwise, in the underlying graph, the lower and upper rays will not belong to the same end. It is not at this moment clear what the implications of this are. With this being said, this framework allows us to consider more complex cycles and elements of the cycle space than have been discussed in this paper, such as those found in [2].

6 Conclusion

Prior work in the graphical approach to the regress problem yields distinct results but not a clear understanding of what separates infinite cycles from infinite linear chains. Through the lens of formal learning theory, we are able to better clarify the nature of this conceptual difficulty. By adopting the approach to infinite cycles introduced in section 4, we are able to preserve intuitions about infinite cycles and develop a criterion between infinite cycles and infinite linear chains based on directed degrees. Though the present methods do not generalize to arbitrary infinite graphs, they begin to illuminate infinite cycles and, in turn, increase our understanding of justification structures.

There are at least two interesting directions for future research. The first is to explore identifying infinite cycles from a learning theory perspective that goes beyond the approach presented in section three. The second is to explore the consequences of more complex cycle structures for the regress problem. Note that the conceptual machinery introduced in section four by no means illustrates the depth of the topological approach to infinite cycles.

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